Effect of dynamic wind force on structures using spectral approaches with complex modal analysis

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ABSTRACT

The dynamic wind loads are considered in terms of equivalent static wind load based on gust-effect factor in most of the current design practices. In ASCE 7-05, the equivalent static wind load is based on the gust-effect factor \(G_f\) which corresponds with the displacement at the highest point of the structure for flexible or dynamically sensitive structures. Gust-effect factors are calculated directly from background and resonance response by spectral method using their real modal properties. The main focus of the present investigation is on the influence of non-proportional viscoelastic damping matrix (for independent-frequency case system) and symmetrical stiffness and damping matrices (frequency-dependent case system) on the spectral calculation. A new transfer function is introduced in complex modal analysis or spectral stochastic response analysis. The case study is clearly demonstrated with numerical calculations showing the accuracy and high performance in this method.

INTRODUCTION

The response of linear elastic flexible system under the dynamic wind load (DWL) has been studied in the past in order to determine the peak dynamic response (PDR) \( \tilde{r} = \bar{r} + g\sigma_r \), where, \( \bar{r} \) is the mean or time average response, \( g \) is the statistical peak factor and \( \sigma_r \) is the standard deviation of the response (SDOR), [1], [2,3,4], [5]. The peak dynamic response can be calculated directly from DWL or indirectly from the equivalent static wind load (ESWL) [6]. In both cases, the damping matrix of the structure is one of the most difficult and important factor to be considered because of the nature of non-proportional viscous property and its significant influence on PDR.

The free vibration of multi-degree-of-freedom (MDOF) with non-proportional damping matrix was developed mathematically by [7]. An improved approach was expressed by using a new coordination \( \{W(t)\} = \{y(t)\}; \{\dot{y}(t)\} \) in the governing equations of motion. The coupled modal problem becomes uncoupled due to the appearance of complex eigenvectors and eigenvalues of the system. Another mathematical methods were presented by [8] and [9]. One could find that these last methods, in general, are difficult to apply on the problem of MDOF system under ambient excitation.

The linear vibration system under random excitation is considered by using the approximate normal mode method presented by [10]. The coupled equation of motion becomes uncoupled based on the utilization of the fictitious damping ratio \( \tilde{\xi}_i \). In effect, this approximation is acceptable when the eigenvalues are well separated.
The vibration of multi-degree-of-freedom under wind loading is analyzed deeply. It is found that the numerical work becomes very cumbersome when the dynamic effects are considered in time domain, [11], [12]. The problem demands even more computational effort when the aeroelastic wind force is accounted for [13]. The estimated response using spectral method is preferred in this case [14], [15].

In the present investigation, the spectral method is summarized and the development of this method for a non-proportional damping system under dynamic wind force is presented. A new transfer function \( H(i\omega) \) is introduced into the power spectral density (PSD) response function. This method is based on spectral method presented in [16]. At the end of the paper, some numerical examples are presented to illustrate the effectiveness of the new approach in structure design.

**THEORETICAL BACKGROUND**

Under quasi-stationary flow wind load, the equation of motion of N-degrees of freedom (N-DOF) damped system with viscous damping can be written in matrix form as:

\[
\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}(\omega)\dot{\mathbf{y}}(t) + \mathbf{K}(\omega)\mathbf{y}(t) = \mathbf{f}_b(t)
\]

where \( \mathbf{y}(t) = [y_1(t), \ldots, y_N(t)] \) is the Lagrangian displacement vector; \( \mathbf{M}, \mathbf{K}(\omega) \) and \( \mathbf{C}(\omega) \) are, respectively, the mass matrix, the stiffness matrix and the damping matrix of the structure. The vector \( \mathbf{f}_b(t) = [f_{b1}(t), \ldots, f_{bN}(t)] \) is the aerodynamic wind force vector, presented by its cross power spectral density (XPSD) matrix \( \mathbf{P}_f(\omega) = [P_{f_{j,m}}(\omega)]_{N,N} = [P_{f_{j,m}}^{*}(\omega)]_{N,N}, \forall j,m = 1, \ldots, N \). The function \( P_{f_{j,m}}^{*}(\omega) \) is referred to as the one-sided XPSD of \( f(t) \) between the two points \( j \) and \( m \) of the structure. These factors were discussed in detail in [14] and [17]. In the frequency domain, Eq. (1) could be written as:

\[
[-\omega^2\mathbf{M} + i\omega\mathbf{C}(\omega) + \mathbf{K}(\omega)]\mathbf{Y}(\omega) = \mathbf{F}_b(\omega)
\]

where \( \mathbf{Y}(\omega) \) and \( \mathbf{F}_b(\omega) \) are the fourier transforms of \( \mathbf{y}(t) \) and \( \mathbf{f}_b(t) \). Therefore, the displacement vector is:

\[
\mathbf{Y}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}(\omega) + \mathbf{K}(\omega)\right]^{-1}\mathbf{F}_b(\omega) = \mathbf{H}(\omega)\mathbf{F}_b(\omega)
\]

Its XPSD function is:

\[
\mathbf{P}_y(\omega) = \mathbf{H}(\omega)\mathbf{P}_f(\omega)\mathbf{H}^*(\omega)
\]

where “*” denotes the conjugate operation. The standard deviation of displacement for the \( k \) th degree of freedom could be calculated directly by:

\[
\sigma_{y_k} = \sqrt{\int_0^\infty P_{y_k}(\omega)d\omega}
\]

This method is easy to use, but it places a high demand on computational effort because of the operation \( \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}(\omega) + \mathbf{K}(\omega)\right]^{-1} = \mathbf{H}(\omega) \) at each discrete frequency. For a frequency-independent system, two other spectral methods, which, correspond to the proportional or non-proportional nature of damping matrix \( \mathbf{C} \) are preferred. The development of a spectral method for a frequency-dependent system is presented in following subsection.
SPECTRAL METHOD FOR A FREQUENCY-INDEPENDENT SYSTEM

In this subsection, it is supposed that the stiffness matrix and the damping matrix are both symmetrical independent on the frequency \( \omega \). Eq. (1) becomes:

\[
\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\mathbf{y}(t) + \mathbf{K}\mathbf{y}(t) = \{f_b(t)\}
\]

(6)

System with proportional damping matrix

The damping matrix \( \mathbf{C} = \alpha_m\mathbf{M} + \alpha_k\mathbf{K} \) is assumed to be Rayleigh damping. Eq. (6) is usually solved by the principal transformation law:

\[
y_k(t) = \sum_{j=1}^{N} \phi_{kj} Z_j(t)
\]

(7)

where \( \{\phi_j\} = \{\phi_{j1}; \phi_{j2}; \ldots; \phi_{jN}\} \) is the structural eigenvector of mode \( j \)th and \( \{Z(t)\} = \{Z_1(t); \ldots; Z_N(t)\} \) is the vector of the principal coordinates representing the image of \( \{y(t)\} \) in the structural principal space. By substitution of Eq. (7) into Eq. (6), the coupled Eq. (6) becomes uncoupled due to the proportional nature of the damping matrix:

\[
\ddot{Z}_j(t) + 2\omega_j\xi_j \dot{Z}_j(t) + \omega_j^2 Z_j(t) = \frac{F_j(t)}{M_j}, \forall j = 1, \ldots, N
\]

(8)

where \( \omega_j \) and \( \xi_j \) are the natural frequency and the damping ratio of the \( j \)th mode, respectively.

For a response \( r_k(t) \) for the \( k \)th degree of freedom, Eq. (7) becomes:

\[
r_k(t) = \sum_{j=1}^{N} b_{kj} Z_j(t)
\]

(9)

Values \( b_{kj} \) are obtained by standard methods of analysis [16]. The power spectral density (PSD) \( P_k(\omega) \) and the variance \( \sigma_k^2 \) of the response \( r_k(t) \) are calculated directly by:

\[
\sigma_k^2 = \int_0^\infty P_k(\omega)d\omega = \int_0^\infty \sum_{j=1}^{N} \sum_{m=1}^{N} b_{kj} b_{km} h_j(-i\omega) h_m(i\omega) P_{F_{j,m}}(\omega)d\omega
\]

(10)

where \( h_j(-i\omega) \) and \( h_m(i\omega) \) are transfer functions:

\[
h_j(\pm i\omega) = \frac{1}{K_j[1 \pm 2i\xi_j(\omega\omega_j) - (\omega\omega_j)^2]}
\]

(11)

System with non-proportional damping matrix

The approximate method for using non-proportional damping matrix ignores the off-diagonal coupling coefficients of the modal damping matrix \( c = \Phi^T C \Phi \) then solve the resulting uncoupled Eq. (8) as a typical mode superposition analysis or by spectrum method presented above [16]. However, there is no mathematical theory or sufficient experimental evidence showing why damping in a physical system should be described by proportional damping [19]. The results presented in examples at the end of the paper indicate that this approximation is not applicable in some cases. In addition, [20] presented an available method for identifying a system with non-proportional damping matrix. Therefore, the development of spectrum method...
using directly non-proportional damping matrix is necessary. Considering the general situation where the damping matrix $C$ is not necessarily Rayleigh damping $C = \alpha_m M + \alpha_k K$. Supposing that $\{y(t)\} = \{\theta_j\}e^{s_jt}$ and a new state vector $\{W(t)\} = \begin{bmatrix} \{y(t)\} \\ \{\dot{y}(t)\} \end{bmatrix}_{2N,1}$ is:

$$\{W(t)\} = \begin{bmatrix} \{\theta_j\} \\ s_j \{\theta_j\} \end{bmatrix} e^{s_jt} = \{\theta_j\}_{2N,1} e^{s_jt}$$

Eq. (6) becomes:

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \{\ddot{W}(t)\} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \{W(t)\} = \begin{bmatrix} \{f_h(t)\} \\ \{0\} \end{bmatrix}$$

(13)

It clearly indicated that $(s_j, s_{j+N})$ are $N$ pairs of conjugate eigenvalue which are solutions of the characteristic polynomial $\text{det}[S[A] + [B]] = 0$ and $(\{\theta_j\}, \{\theta_{j+N}\})$ are $N$ pairs of conjugate associated eigenvectors having $2N$ components [7]. $\{\theta_j\}$ is calculated from equation $[s_j[A] + [B]]\{\theta_j\} = \{0\}$ and its form is $\{\theta_j\} = \begin{bmatrix} \{\theta_j\} \\ s_j \{\theta_j\} \end{bmatrix}_{2N,1}$, $\forall j = 1, \ldots, 2N$.

Make $\{W(t)\} = \Theta\{Z(t)\}_{2N,1}$, where $\{Z(t)\}_{2N,1}$ is called a new general coordinate vector and the full matrix $\Theta = [\theta_{jm}]_{2N,2N}$ contains $2N,2N$ complex components. Eq. (13) is now:

$$\text{diag}[a_j] \{\dot{Z}(t)\} + \text{diag}[b_j] \{Z(t)\} = \Theta^T \begin{bmatrix} \{f_h(t)\} \\ \{0\} \end{bmatrix}$$

(14)

where $\text{diag}[a_j]_{2N,2N} = \Theta^T [A] \Theta$ and $\text{diag}[b_j]_{2N,2N} = \Theta^T [B] \Theta$. It is easy to show that $(a_j, a_{j+N})$ and $(b_j, b_{j+N})$ are $2N$ pairs of conjugate. The solutions for $2N$ differential Eq. (14) order one:

$$(Z(t))_{2N,1} = \begin{bmatrix} \int_{-\infty}^{\infty} F_j(\tau) h_j(t - \tau) d\tau \end{bmatrix}_{2N,1}$$

(15)

where $h_j(t)$ is the impulsive -response function and $F_j(\tau) = \{\theta_j\}^T \{f(\tau)\}$ is the complex modal force, $\forall j = 1, \ldots, 2N$. Take a general equation at mode $k$, $\forall k = 1, \ldots, 2N$:

$$Z_k(t) = \int_{-\infty}^{\infty} F_k(\tau) h_k(t - \tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(i\omega) F_k(\omega) e^{i\omega t} d\omega$$

(16)

Here $H_k(i\omega) = \int_{-\infty}^{\infty} h_k(t) e^{-i\omega t} dt$ and $F_k(\omega) = \int_{-\infty}^{\infty} F_k(t) e^{-i\omega t} dt$ are Fourier transforms of $h_k(t)$ and $F_k(t)$. Note that the vector turbulent wind loads $\{f_h(t)\}$ contains $N$ stationary gaussian processes having a zero mean value, the fourier series representation could be used:

$$F_k(t) = \sum_{n=-\infty}^{\infty} \hat{F}_k(n) = \sum_{n=-\infty}^{\infty} F_k(\omega_n) e^{i\omega_n t}$$

(17)

The modal displacement for the $k$th mode is:

$$Z_k(t) = \sum_{n=-\infty}^{\infty} Z_{nk}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{a_k \omega_n i + b_k} F_k(\omega_n) e^{i\omega_n t}$$

(18)

or:

$$Z_k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a_k \omega i + b_k} F_k(\omega) e^{i\omega t} d\omega$$

(19)
From Eq. (16) and Eq. (19), one has:

\[ H_k(i\omega) = \frac{1}{a_ki\omega + b_k} = \int_{-\infty}^{\infty} h_k(t)e^{-i\omega t} dt \] (20)

and:

\[ h_k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_k(\omega)e^{i\omega t} d\omega = \begin{cases} \frac{b_k}{a_k}, & \forall t \geq 0 \\ 0, & \forall t < 0 \end{cases} \] (21)

Keeping in view that \( a_k \) and \( b_k \) are functions of \( \Theta \), \([A]\) and \([B]\) which contain all the properties of the system, one can estimate the response using directly the impulsive-response function \( h_k(t) \) in the complex mode superposition analysis, or indirectly the transfer function \( H_k(i\omega) \) in the spectral stochastic response analysis.

- **Complex mode superposition analysis**

Replacing Eq. (21) into Eq. (15), the general coordinate vector \( \{Y(t)\}_{2N,1} \) of the system is:

\[ \{Z(t)\}_{2N,1} = \left\{ \int_{-\infty}^{\infty} F_j(\tau) \left( \frac{1}{a_j} e^{\frac{b_j(\tau)}{a_j}} \right) d\tau \right\}_{2N,1} \] (22)

So the displacement vector \( \{y(t)\}_{N,1} \) is:

\[ \{y(t)\}_{N,1} = [\Theta_{N,2N}^{ub}] \{Z(t)\}_{2N,1} \] (23)

where, \( [\Theta_{N,2N}^{ub}] = [\theta_{jm}]_{N,2N} \) is the upper-half of the full complex eigenmatrix \( \Theta \). The other response vector \( \{r(t)\}_{N,1} = \sum_{j=1}^{2N} [B_{kj}]Z_j(t) \) of the structure is calculated similarly by:

\[ \{r(t)\}_{N,1} = [B_{j}]_{N,2N} \{Z(t)\}_{2N,1} \] (24)

where \( B_{kj} \) is obtained by standard methods of analysis in using a vector elastic force \( \{f_s(t)\} = K\{y(t)\} \):

\[ \{f_s(t)\} = K[\Theta_{N,2N}^{ub}] \{Z(t)\}_{2N,1} \] (25)

- **Spectral stochastic response analysis**

In [16], the PSD of the response was analyzed mathematically for a proportional damping system. The following paragraph presents the development of this method for the non-proportional damping system. The results presented in numerical exemples indicate the reliability of this expansion.

From Eq. (15) and Eq. (23), the equation of displacement for the \( k \) th degree of freedom is:

\[ y_k(t) = \sum_{j=1}^{2N} \theta_{kj}Z_j(t) = \sum_{j=1}^{2N} \theta_{kj} \int_{-\infty}^{\infty} F_j(\tau)h_j(t-\tau) d\tau \] (26)

The PSD of the displacement \( y_k(t) \) is defined from its autocorrelation function:

\[ P_{y_k}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{y_k}(\tau)(\omega) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} R_{y_k}(\tau)exp(-i\omega \tau) d\tau \] (27)
where $F_{R_k}(\tau)(\omega)$ is the Fourier transform of the autocorrelation function $R_k(\tau)$ for $y_k(t)$:
\[
R_k(\tau) = E \left( \sum_{j=1}^{2N} \sum_{m=1}^{2N} \theta_{j_m} Z_j(t) Z_m(t + \tau) \right) \tag{28}
\]
where $E(x(t))$ denotes the mean value of the discrete random variable $x(t)$. Replacing $Z_j(t)$ and $Z_m(t + \tau)$ of Eq. (15) into Eq. (28) and noting that only $F_j(\tau_j)$ and $F_m(\tau_m)$ change across the ensemble, one could obtain:
\[
R_k(\tau) = \sum_{j=1}^{2N} \sum_{m=1}^{2N} \theta_{j_m} \theta_{k_m} (\eta_j) h_m(\eta_m) R_{F_{j,m}}(\tau - \eta_m + \eta_j)d\eta_j d\eta_m \tag{29}
\]
where $\eta_j = t - \tau_j$ et $\eta_m = t + \tau - \tau_m$. Replacing $R_k(\tau)$ from Eq. (34) into Eq. (32) and changing variable $\gamma = \tau - \eta_m + \eta_j$, Eq. (27) becomes:
\[
P_{y_k}(\omega) = \sum_{j=1}^{2N} \sum_{m=1}^{2N} \theta_{j_m} H_j(-i\omega) H_m(i\omega) P_{F_{j,m}}(\omega) \tag{30}
\]
with:
\[
P_{F_{j,m}}(\omega) = \frac{1}{2\pi} \lim_{\gamma \to \infty} \int_{-\infty}^{\infty} R_{F_{j,m}}(\gamma) \exp(-i\omega \gamma) d\gamma \tag{31}
\]

The transfer functions $H_j(i\omega)$ and $H_m(i\omega)$ are calculated by Eq. (20) and $H_j(-i\omega)$ and $H_j(i\omega)$ are conjugate ($H_j(-i\omega) = \frac{1}{-a_j\omega_k + b_j}$). The variance of the displacement is:
\[
\sigma^2_{y_k} = R_{y_k}(0) = \int_{-\infty}^{\infty} P_{y_k}(\omega) d\omega \tag{32}
\]

For the other response $r_k(t)$ at the $k$th degree of freedom of the structure, its PSD becomes:
\[
P_{r_k}(\omega) = \sum_{j=1}^{2N} \sum_{m=1}^{2N} B_{j_k} B_{m_m} H_j(-i\omega) H_m(i\omega) P_{F_{j,m}}(\omega) \tag{33}
\]
and its variance $\sigma^2_{r_k} = \int_{-\infty}^{\infty} P_{r_k}(\omega) d\omega$.

**Spectral method for a frequency-dependent system**

In this paragraph, a more general case is considered with the symmetrical stiffness matrix and the damping matrix both dependent on the frequency $\omega$.

From Eq. (1), the new system under turbulent wind load could be presented as:
\[
\begin{bmatrix}
C(\omega) & M \\
M & 0
\end{bmatrix} \{\dot{W}(t)\} + \begin{bmatrix}
K(\omega) & 0 \\
0 & -M
\end{bmatrix} \{W(t)\} = \begin{bmatrix}
\{f_i(t)\} \\
\{0\}
\end{bmatrix} \tag{34}
\]
where the state vector $\{W(t)\} = \begin{bmatrix} \{\theta_j(\omega)\} \\ s_j(\omega) \{\theta_j(\omega)\} \end{bmatrix} e^{s_j(\omega)t} = \{\theta_j(\omega)\}_{2N,1} e^{s_j(\omega)t}$. The complex eigenvalues $s_j(\omega)$ and eigenvectors $\theta_j(\omega)$ are calculated from equations $\det[s_j(\omega)[A(\omega)] + [B(\omega)]] = 0$ and $[s_j(\omega)[A(\omega)] + [B(\omega)] \{\theta_j(\omega)\} = \{0\}$, $\forall \, j, J = 1, \ldots, 2N$. 


Take \( \{W(t)\} = \Theta(\omega)\{Z(t)\}_{2N,1} \) where the full matrix \( \Theta(\omega) = [\theta_{jm}(\omega)]_{2N,2N} \) contains \( 2N \cdot 2N \) complex frequency-dependent components. Eq. (34) is now:

\[
diag[a_j(\omega)]\{\dot{Z}(t)\} + diag[b_j(\omega)]\{Z(t)\} = \Theta^T(\omega) \begin{bmatrix} \{f_k(t)\} \\ \{0\} \end{bmatrix} \tag{35}
\]

where \( diag[a_j(\omega)] = \Theta^T(\omega)[A(\omega)]\Theta(\omega) \) and \( diag[b_j(\omega)] = \Theta^T(\omega)[B(\omega)]\Theta(\omega) \). By using the Fourier transform, Eq. (35) becomes:

\[
Z_k(\omega) = H_k(i\omega)F_k(\omega) = \frac{1}{a_k(\omega)\omega + b_k(\omega)} \int_{-\infty}^{\infty} \{\theta_k(\omega)\}^T \{f_k(t)\} e^{-i\omega t} dt \tag{36}
\]

The displacement \( y_k(t) \) for the \( k \) th degree of freedom is:

\[
y_k(t) = \sum_{j=1}^{2N} \theta_j(\omega) Z_j(t) = \sum_{j=1}^{2N} \theta_j(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} H_j(i\omega)F_j(\omega) e^{i\omega t} d\omega \tag{37}
\]

Using the same approach presented in subsection above, the PSD of the displacement \( y_k(t) \) is:

\[
P_{y_k}(\omega) = \sum_{j=1}^{2N} \sum_{m=1}^{2N} \theta_j(\omega) \theta_{jm}(\omega) H_j(-i\omega)H_m(i\omega) P_{Fj,m}(\omega) \tag{38}
\]

with

\[
P_{Fj,m}(\omega) = \{\theta_j(\omega)\}^T P_j(\omega) \{\theta_m(\omega)\} \tag{39}
\]

Therefore, the variance of the displacement \( y_k(t) \) is \( \sigma_{y_k}^2 = \int_0^\infty P_{y_k}(\omega) d\omega \).

**Numerical Examples**

Two numerical examples are presented in this study. The turbulent wind velocities \( v(t) \) which create the load on each mass is generated from PSD \( P_v(\omega) \) [14]:

\[
P_v(\omega) = 2\pi \frac{200z}{\bar{V}(z)} \left[ 1 + 50 \frac{\omega z}{\bar{V}(z)} \right] \tag{40}
\]

here, \( \bar{V}(z)_{z=10} = 25m/s \) is mean wind speed at level \( z(m) \) above ground and \( v_s = 1.892(m/s) \) is the shear friction velocity. The XPSD of the wind velocity is considered in two cases: linear XPSD \( P_{v,j,m}^\alpha(\omega) = \alpha_j \alpha_m P_v(\omega) \) where \( v_{j,m}(t) = \alpha_j v(t) \) are the wind velocities at two masses \( j \) and \( m \) (example 01) and non-linear XPSD (example 02):

\[
P_{v,j,m}^\gamma(\omega) = \gamma_{v,j,m}(\omega) \sqrt{P_{v,j}(\omega) P_{v,m}(\omega)} \tag{41}
\]

with:

\[
\gamma_{v,j,m}(\omega) = \exp \left\{ -\frac{\omega}{2\pi} \sqrt{\sum_{r} C_{rv}(r_j - r_m)^2} \right\} \right\} \frac{1}{2} \left[ \bar{V}_j + \bar{V}_m \right] \tag{42}
\]

where \( C_{rv} \) is the exponential decay coefficient of the turbulence component \( v \) in the direction \( r = x, y \) and \( z \), respectively. In the second example, \( C_{sv} = 10.5 \) and \( C_{sv} = C_{sv} = 0 \).
It is assumed that the time series of wind force applied on \( j \)th mass \( f_j(t) \equiv A v_j(t) \) where \( A \) is a constant, in \((N s/m)\). The PSD of wind force is \( P_f(\omega) = A^2 P_{v_j}(\omega) \) and its XPSD is \( [P_f(\omega)] = [P_{f_{j,m}}(\omega)] = A^2 [P_{v_{j,m}}(\omega)] \).

**Example 01**

In order to evaluate the presented method, the 2-DOF asymmetrical vibration system shown in Fig. 1 is considered as a first numerical example. The system has the mass matrix \( M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and the stiffness matrix \( K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \). By changing the values \( k_1 \) and \( k_2 \), two natural frequencies of the system are well separated or close. The case \((k_1; k_2) = (1.5; 0.1)\) is presented here.

The damping matrix is \( C = 10^5 \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \). By changing the values \( c_1 \) and \( c_2 \) of this matrix, one can obtain a proportional or non-proportional damping matrix \( C \). Several pairs \((c_1; c_2)\) are considered.

**Step-by-step solution approach**

The Newmark constant-average-acceleration method is adopted. The algorithm presented in [21] with \((\delta; \alpha) = (0.5; 0.25)\) and \( T = N \Delta t = 32768 \times 0.125 = 4096 \) is used. The five loading vectors \( \{R(t)\} = \{f_1(t); 0.5 f_1(t)\} \) are generated from the PSD function of Eq. (40). The mean of five standard deviations of displacement of second degree of freedom is calculated from the last \( N/2 \) values of each time series:

\[
\sigma_{\text{y}2}^\text{NM} = \sqrt{\frac{\sum_{j=1}^{5} \int_{T/2}^{T} y_2^2(t_j) dt_j}{T/2}}
\]  

**Integral approach**

Using Eq. (5), the SDOR of the displacement \( y_2(t) \) is:

\[
\sigma_{\text{y}2}^\text{ln} = \sqrt{\int_0^{\omega_2} P_y^{(2,2)}(\omega) d\omega}
\]

where

\[
P_y(\omega) = \left[ -\omega^2 M + i \omega C + K \right]^{-1} P_f(\omega) \left[ -\omega^2 M + i \omega C + K \right]^{-1} \]

**Quasi-proportional approach**

When the damping matrix \( C \) is non-proportional, the modal damping matrix \( c = \Phi^T C \Phi \) is non-diagonal. Nevertheless, an approximate solution may be obtained by ignoring the off-diagonal coupling coefficients of matrix and then solving the resulting uncoupled equations as a typical mode superposition analysis. This approach is named quasi-proportional (QP) approach in this study. The SDOR can be calculated from equation:
\[ \sigma_{y_2}^{QP} = \sqrt{ \sum_{j=1}^{2} \sum_{m=1}^{2} \phi_j \phi_m \int_0^\infty h_j(-i\omega)h_m(i\omega)P_{F_{j,m}}(\omega) \, d\omega } \]  \hspace{1cm} (46)

where, \( P_{F_{j,m}}(\omega) = \phi_j^T P_f(\omega) \phi_m \)

**Approximation quasi-proportional approach**

By ignoring the off-diagonal coupling coefficients of the modal damping matrix \( c = \Phi^T C \Phi \), for lightly damped systems with well separated modal frequencies, the SDOR can be calculated by another quasi-proportional approximation (QPA) approach:

\[ \sigma_{y_2}^{QPA} = \sqrt{ \sum_{j=1}^{2} \sum_{m=1}^{2} \phi_j^2 K_j \int_0^\infty P_{F_j}(\omega) \, d\omega + \sum_{j=1}^{2} \phi_j^4 \pi \omega_j P_{F_j}(\omega_j) } \]  \hspace{1cm} (47)

- **Non-proportional approach**

The standard deviation of displacement \( y_2(t) \) is:

\[ \sigma_{y_2}^{NP} = \sqrt{ \sum_{j=1}^{4} \sum_{m=1}^{4} \theta_j \theta_m \int_0^\infty H_j(-i\omega)H_m(i\omega)P_{F_{j,m}}(\omega) \, d\omega } \]  \hspace{1cm} (48)

where \( P_{F_{j,m}}(\omega) = \theta_j^T P_f(\omega) \theta_m \), and \( \{ \theta_j \} \), \( \{ \theta_m \} \) are \( j \)th, \( m \)th column vectors of the upper half of the eigenvectors matrix \( \Theta_{2,4}^{uh} \).

**Standard deviation comparison**

In this case, \( (k_1;k_2) = (1.5;0.1) \); \( A=1 \) and the wind forces \( f_1(t) = 2f_2(t) \) apply on two masses \( m_1 \) and \( m_2 \), respectively. So its XPSD is \( [P_y(\omega)] = A^2 [P_r(\omega)] = P_r(\omega) \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix} \).

The stiffness matrix is \( K = 141750 \begin{bmatrix} 1.6 & -0.1 \\ -0.1 & 1.6 \end{bmatrix} \). The two natural frequencies of the undamped system are \( (\omega_1;\omega_2) = (1.458;1.55) \) (\( \text{rad/s} \)).

When \( (c_1;c_2) = (0.5;0.1) \), the upper half of the eigenvectors matrix \( \Theta \) of the system is:

\[ \Theta_{2,4}^{uh} = \begin{bmatrix} -0.29 + 0.40i & -0.25 + 0.39i & -0.29 - 0.40i & -0.25 - 0.39i \\ -0.28 + 0.40i & 0.25 - 0.39i & -0.28 - 0.40i & 0.25 + 0.39i \end{bmatrix} \]  \hspace{1cm}

The standard deviation of displacement \( y_2(t) \) is calculated by five approaches presented above. Fig. 2 presents this response in time series and its PSD by four spectral approaches. The comparison of these results are shown in Table 1.

---

**Figure 1: 2-DOF viscous linear vibration system**
Figure 2: Displacement $y_2(t)$ when $(k_1; k_2)=(1.5; 0.1)$ and $(c_1; c_2)=(0.5; 0.1)$

Table 1: Standard deviation comparison $\sigma_{y_2(t)}$

<table>
<thead>
<tr>
<th>$c_1; c_2$</th>
<th>$\sigma_{QPA}$</th>
<th>$\text{Err} \to$</th>
<th>$\sigma_{\text{In}}$</th>
<th>$\neq \to$</th>
<th>$\sigma_{\text{NP}}$</th>
<th>$\neq \to$</th>
<th>$\sigma_{\text{NP}}$</th>
<th>$\neq \to$</th>
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<tr>
<td></td>
<td>$10^{-5}$ m</td>
<td>%</td>
<td>$10^{-5}$ m</td>
<td>%</td>
<td>$10^{-5}$ m</td>
<td>%</td>
<td>$10^{-5}$ m</td>
<td>%</td>
<td>$10^{-5}$ m</td>
</tr>
<tr>
<td>1.5; 0.1</td>
<td>4.71</td>
<td>10.36</td>
<td>4.27</td>
<td>-0.38</td>
<td>4.29</td>
<td>4.5</td>
<td>4.48</td>
<td>-35.9</td>
<td>2.87</td>
</tr>
<tr>
<td>1.5; 0.5</td>
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<td>35.47</td>
<td>3.46</td>
<td>0.02</td>
<td>3.46</td>
<td>2.9</td>
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<td>-16.9</td>
<td>2.96</td>
</tr>
<tr>
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<td>29.86</td>
<td>3.61</td>
<td>7.46</td>
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<td>2.2</td>
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<td>3.01</td>
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<tr>
<td>0.5; 0.1</td>
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<td>11.18</td>
<td>4.87</td>
<td>-0.61</td>
<td>4.90</td>
<td>5.1</td>
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<tr>
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<td>4.25</td>
<td>4.4</td>
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<tr>
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<td>29.40</td>
<td>4.15</td>
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<td>4.16</td>
<td>4.2</td>
<td>4.34</td>
<td>-10.8</td>
<td>3.87</td>
</tr>
<tr>
<td>0.1; 0.1</td>
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<td>11.07</td>
<td>7.56</td>
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<td>7.64</td>
<td>6.5</td>
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<td>4.45</td>
<td>4.8</td>
<td>4.66</td>
<td>-34.8</td>
<td>3.04</td>
</tr>
</tbody>
</table>

$C_{\text{test}}=0.1M + 0.1K$

**EXAMPLE 02**

The 4-DOF vibration system sketched in Fig. 4 is considered as a second numerical example. The damping matrix $C(\omega)$, the stiffness matrix $K(\omega)$ and the mass matrix $M$ are:

$$C = 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c_1 \omega^{2\beta} & -c_1 \omega^{2\beta} & 0 \\ 0 & -c_1 \omega^{2\beta} & c_1 \omega^{2\beta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = 141750 \begin{bmatrix} 0 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}$$
and $M = 10^5 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By changing the value $\beta$, several responses are compared.

The XPSD of wind force which applies on four masses is $[P_f(\omega)] = [P_{f,j,m}^T(\omega)] = A^T[P_{f,j,m}^T(\omega)] = A^T P_v(\omega)[\gamma_{v,j,m}(\omega)]$ where $P_v(\omega)$ is presented in Eq. (40)

and $\gamma_{v,j,m}(\omega)$ is calculated by Eq. (42): $\gamma_{v,j,m}(\omega) = \exp \left( \frac{-10.5 \frac{\omega}{2\pi} |y_j - y_m|}{25} \right)$.

The interesting response in this example is the displacement $y_4(t)$ for the 4th mass. Its standard deviation is calculated by two similar spectral approaches presented in example 01.

**Integral approach**

Using Eq. (5), the SDOR of the displacement $y_4(t)$ is:

$$\sigma_{y_4}^{\text{ln}} = \sqrt{\int_0^\infty P_{y}^{(4;4)}(\omega)d\omega}$$

where

$$P_y(\omega) = H(\omega)P_f(\omega)H^*(\omega)$$

$$H(\omega) = \left[ -\omega^2 M(\omega) + i\omega C(\omega) + K(\omega) \right]^{-1}$$

**Non-proportional approach**

The standard deviation of the displacement $y_4(t)$ is:

$$\sigma_{y_4}^{\text{NP}} = \sqrt{\int_{j=1}^{m} \sum_{j=1}^{m} \theta_{4,j}(\omega)\theta_{4,m}(\omega) \int_0^\infty H_j(-i\omega)H_m(i\omega)P_{f,j,m}(\omega)d\omega}$$

where $P_{f,j,m}(\omega) = \{\theta_j(\omega)\}^T [P_f(\omega)]\{\theta_m(\omega)\}$, and $\{\theta_j(\omega)\}$, $\{\theta_m(\omega)\}$ are $j$th, $m$th column vectors of the upper half of the eigenvectors matrix $\Theta_{+4}^{\text{uh}}(\omega)$.

**Standard deviation comparison**

In this example, $(k_1; k_2; k_3; k_4) = (4;3;2;1)$; $c_1 = 2$ and $A=1$. The standard deviation of the displacement $y_4(t)$ is calculated by two approaches presented above. Fig. 4 presents its PSD by two spectral methods in the case $\beta = 0.1$. The comparison of these results is shown in Table 2.
The differences between the two spectral approaches are very small. It means that the development of the method presented in subsection above is reliable.

**CONCLUSION AND RECOMMENDATION**

For a viscous linear elastic of MDOF system under dynamic wind loads, some conclusions are presented here:

- The development of spectral methods using a non-proportional damping matrix of the system (independent frequency or not) is performed in this study. The difference between the responses of the non–proportional approach and step–by–step approach or integral approach is very low. The calculation of any results by spectral method is faster and more reliable than by dynamic transient methods.

- The errors between quasi-proportional and non-proportional or step-by-step approaches are considerable when the natural frequencies of the system are
closed. The displacement calculated from its background and resonant components is always over-estimated in this example.

- The utilization of spectral method with quasi-proportional damping matrix is not recommendable. It may be acceptable when the natural frequencies of the system are well separated and the structure is lightly damped.

- The both aerodynamic and aeroelastic wind force should be analyzed by non–proportional approach with frequency dependent properties of the system. The expansion of this approach for the asymmetrical frequency dependent system much be studied in the next research.

REFERENCES


